

Jump-defects in the nonlinear Schrödinger model and other non-relativistic field theories

E Corrigan¹ and C Zambon²

¹ Department of Mathematics, University of York, York YO10 5DD, U.K.

² Yukawa Institute for Theoretical Physics, University of Kyoto, Kyoto 606-8502, Japan

E-mail: ec9@york.ac.uk and zambon@yukawa.kyoto-u.ac.jp

Abstract. Recent work on purely transmitting ‘jump-defects’ in the sine-Gordon model and other relativistic field theories is extended to non-relativistic models. In all the cases investigated the defect conditions are provided by ‘frozen’ Bäcklund transformations and it is also shown via a Lax pair argument how integrability will be preserved in the presence of this type of defect. Explicit examples of the scattering of solitons by defects are given, and bound states associated with ‘jump-defects’ in the nonlinear Schrödinger model are described. Although the nonlinear Schrödinger model provides the principal example, some results are also presented for the Korteweg de Vries and modified Korteweg de Vries equations.

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1. Introduction

The study of impurities, or defects, has a lengthy history especially in the context of condensed matter physics (see for example the lecture notes by Saleur [1]). In the context of integrable field theories of various types there has been interest in studying defects both from a classical and a quantum point of view, either theoretically or from the applications perspective. In the quantum domain there are issues surrounding the extent to which integrability is compatible with the reflection and transmission typical of a defect. The pioneering work of Delfino, Mussardo and Simonetti [2] represents one point of view and more recent work of Mintchev, Ragoucy and Sorba [3] provides another. Typically, adding a δ -impurity to a classically integrable nonlinear field theory destroys its integrability, although there may nonetheless be interesting phenomena associated with defects of this type (see [4] for the behaviour of solitons in the sine-Gordon model with a δ -impurity, or [5] and [6] for studies of some aspects of the nonlinear Schrödinger model with a δ -impurity). On the other hand, there may be circumstances where a different type of defect is able to preserve the property of classical integrability and it is an interesting question to investigate what those circumstances might be. Some years ago, it was noticed that several types of relativistic integrable

field theory permit discontinuities (a type of defect) without the property of classical integrability being destroyed [7]. Among these are free fields, Liouville theory, the sine/sinh-Gordon model and a variety of affine Toda field models (though possibly not all of them [8]). Besides preserving integrability, it was found that the conditions defining the defect allowed not only energy conservation (as would be expected) but also the conservation of a generalised momentum, including a contribution from the defect itself (which was unexpected because of the evident loss of translation invariance). An integrable discontinuity of this type will be referred to as a ‘jump-defect’ in order to distinguish it from a (generally non-integrable) δ -impurity.

The integrable jump-defects have a Lagrangian description and their integrability is ensured by the existence of suitably adapted Lax pairs. The specific form of the jump-defect conditions is quite striking and makes novel use of Bäcklund transformations which are well-known to be a bulk feature of each of the models considered (for example, see [9]). With a single jump-defect the setup is quite straightforward and easily described. One of the simplest examples is provided by the sine-Gordon model, as follows.

The sine-Gordon model in the bulk [10, 11, 12] is specified by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(u_t^2 - u_x^2) - \frac{m^2}{\beta^2}(1 - \cos \beta u). \quad (1.1)$$

A single jump-defect placed at $x = 0$ is then described by altering the Lagrangian in the following manner, denoting the field on the left of the defect by u and the field on the right of it by v . The full Lagrangian consists of pieces from the bulk regions ($x < 0$ and $x > 0$), together with a delta function contribution at $x = 0$. Thus, the new Lagrangian density is given by

$$\mathcal{L} = \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v - \delta(x) \left[\frac{1}{2}(uv_t - vu_t) + \mathcal{B}(u, v) \right], \quad (1.2)$$

with

$$\mathcal{B} = \frac{2m\sigma}{\beta^2} \cos \beta \left(\frac{u+v}{2} \right) + \frac{2m}{\sigma\beta^2} \cos \beta \left(\frac{u-v}{2} \right). \quad (1.3)$$

The form of the additional piece is required by integrability and leads to defect conditions linking the two fields u and v and their derivatives at $x = 0$, the position of the defect. The usual variational principle reveals that the defect conditions constitute a Bäcklund transformation ‘frozen’ at the defect. There are many interesting features of these defects, both from a classical field theory point of view - especially with regard to the behaviour of solitons - and within the quantum context [13]. There is no obstacle to having several defects at different locations (with the same or different parameters). Moreover, they can move and scatter amongst themselves [13]. As mentioned already, a novel feature exhibited by these defects is the manner in which they may exchange both energy and momentum with the fields on either side of the defect location. There is nothing surprising about the energy since time translation is unbroken, but the existence of a conserved - though modified - momentum is quite surprising since translation invariance is certainly lost. In fact, the defect potential (1.3) may be regarded as being

determined by demanding that it be possible to find a conserved generalised momentum functional. The ability to exchange momentum and energy with the fields on either side of it is a defining feature of a jump-defect, and will be used as a tool later in this paper.

As a further example, and to make clear the distinction between a jump-defect and a δ -impurity, consider two free-field situations. The jump-defect can be described as a small field approximation to (1.2, 1.3). In other words,

$$\mathcal{B}(u, v) = -\frac{m\sigma}{4}(u + v)^2 - \frac{m}{4\sigma}(u - v)^2 \quad (1.4)$$

with

$$\mathcal{L}_u = \frac{1}{2}(u_t^2 - u_x^2 - m^2 u^2), \quad \mathcal{L}_v = \frac{1}{2}(v_t^2 - v_x^2 - m^2 v^2). \quad (1.5)$$

On the other hand, in the same notation, a δ -impurity is described by the quadratic Lagrangian,

$$\mathcal{L}_\delta = \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v - \delta(x)\frac{1}{2}[\sigma uv - (u_x + v_x)(u - v)], \quad (1.6)$$

leading to free Klein-Gordon equations in the bulk together with the defect conditions

$$u = v, \quad v_x - u_x = \sigma u, \quad x = 0. \quad (1.7)$$

It is not difficult to check by explicit calculation that the former permits a conserved total momentum functional, including a defect contribution, while the latter does not. There is no requirement for the linear jump-defect to satisfy $u = v$ at $x = 0$, and in general there will be a discontinuity. It is easy to check that the linear jump-defect is purely transmitting, while the δ -impurity both transmits and reflects. The jump-defect does not possess a classical bound state but the δ -impurity does for a suitable range of σ .

Besides the sine/sinh-Gordon model there are other integrable equations which have arisen naturally in special physical systems. Most of these are non-relativistic but might nevertheless allow discontinuities related to jump-defect conditions of frozen-Bäcklund type. One such example is the nonlinear (cubic) Schrödinger equation (NLS); others are the Korteweg-de Vries (KdV) and modified KdV (mKdV) equations.

As already mentioned briefly, defects in the context of integrable field theories have been discussed before in the quantum domain, starting with Delfino, Simonetti and Mussardo [2], and elaborated subsequently by others, including Konik and LeClair [14], and Castro-Alvaredo, Fring and Göhmann [15]. As a consequence of their work it appeared that a defect could not allow simultaneous reflection and transmission while maintaining integrability (in the sense that the reflection and transmission matrices together with the bulk S-matrix should satisfy a set of algebraic compatibility requirements) unless the bulk scattering matrix was independent of rapidity. On the other hand, the sine-Gordon jump-defect is purely transmitting as far as its behaviour with respect to solitons is concerned and evidence was gathered in [13] to support the idea that the pure transmission matrix discovered by Konik and LeClair really describes the quantum version of the sine-Gordon jump-defect. Recently, Mintchev, Ragoucy and Sorba have suggested alternative compatibility relations that might be satisfied between

the transmission and reflection factors and the bulk S-matrix [3]. Using this framework δ -type impurities have been explored in the context of NLS [16].

The purpose of this paper is to demonstrate that jump-defects can be incorporated into the NLS model very naturally without spoiling integrability. Following an analysis of a few of the conserved quantities - actually sufficient to determine the form of the defect conditions - an argument based on a generalised Lax pair is given in sufficient detail to demonstrate how integrability is preserved. Similar but less complete arguments are also given in the context of KdV and mKdV. In all cases, remarks are made concerning the behaviour of solitons as they encounter a jump-defect. To date, the only genuine surprise relative to the sine-Gordon case is provided by the strange (and still mysterious) behaviour of a ‘fast’ soliton in KdV. It remains to be seen what the quantum version of the NLS jump-defect will turn out to be and how it will relate to earlier work, if at all.

2. The nonlinear Schrödinger equation

The nonlinear Schrödinger equation with a cubic interaction term will be taken to be defined by the field equation [11]

$$iu_t + u_{xx} + 2u(\bar{u}u) = 0. \quad (2.1)$$

This may be derived in the bulk using an action principle based on the Lagrangian

$$\mathcal{L} = \frac{i}{2}(\bar{u}u_t - \bar{u}_t u) - |u_x|^2 + \sigma^2 |u|^4. \quad (2.2)$$

A (real, positive) coupling constant σ^2 has been added but then scaled away in the field equation by redefining u . Henceforth it will be ignored. The sign of the cubic term is important for some considerations. For example, the sign chosen in (2.1) is appropriate for a model possessing soliton solutions (corresponding to an ‘attractive’ interaction).

If there is a defect at $x = 0$ then the bulk fields to either side of it will be denoted u and v , and a boundary contribution \mathbf{B} , presumably depending on u, v, u_t and v_t (and possibly spatial derivatives), will need to be added. In other words, the full action will be

$$\mathcal{A} = \int dt \left[\int_{-\infty}^0 dx \mathcal{L}(\mathbf{u}) + \mathbf{B} + \int_0^{\infty} dx \mathcal{L}(\mathbf{v}) \right]. \quad (2.3)$$

The corresponding defect conditions at $x = 0$ are:

$$u_x = \frac{\partial \mathbf{B}}{\partial \bar{u}} - \frac{\partial}{\partial t} \frac{\partial \mathbf{B}}{\partial \bar{u}_t}, \quad v_x = -\frac{\partial \mathbf{B}}{\partial \bar{v}} - \frac{\partial}{\partial t} \frac{\partial \mathbf{B}}{\partial \bar{v}_t}, \quad (2.4)$$

with similar expressions for the conjugate fields. Note that (2.4) would need to be modified if the defect part of the action depended additionally upon the spatial derivatives; for the time being it will be assumed these are absent.

The analogue of energy for NLS is the density

$$\mathcal{E} = |u_x|^2 - |u|^4, \quad (2.5)$$

and, since the system remains time-translation invariant despite adding the defect, the total energy including a defect contribution will be conserved. In the bulk, as a consequence of space-translation invariance, the momentum has the density

$$\mathcal{P} = i(\bar{u}u_x - \bar{u}_xu) \quad (2.6)$$

and it is certainly conserved. However, when there is an impurity the momentum is not expected to be conserved since translation invariance is broken. However, as has been noted before in other cases, for example in the sine-Gordon model, this is not necessarily so. Adding a jump-defect does not spoil momentum conservation since this kind of defect can exchange both energy and momentum with the fields u and v to either side of it. To demonstrate this, note that the total contribution from the fields u and v to the momentum density is given by

$$P = P(u) + P(v) = \int_{-\infty}^0 dx i(\bar{u}u_x - \bar{u}_xu) + \int_0^{\infty} dx i(\bar{v}v_x - \bar{v}_xv) \quad (2.7)$$

and it easy to see that the time derivative of P will depend critically on the defect conditions since

$$P_t = \left(2|u|^4 + 2\bar{u}_xu_x + i(\bar{u}u_t - \bar{u}_tu)\right)_{x=0} - \left(2|v|^4 + 2\bar{v}_xv_x + i(\bar{v}v_t - \bar{v}_tv)\right)_{x=0}. \quad (2.8)$$

To allow conservation of this charge requires that the right hand side of (2.8) should (if possible) be a total time derivative of a functional of the fields evaluated at the jump-defect. Put alternatively, if the momentum, suitably modified, is to be preserved then the defect conditions ought to be chosen to ensure it. This is not quite straightforward to achieve (and the conditions associated with a δ -impurity do not have this property) but, bearing in mind (2.4) the following suggestion works perfectly. It will be seen later that it also fits more generally with the idea of integrability. It is sufficient to take

$$\mathbf{B} = \Omega \left[\frac{i}{2} \frac{\partial}{\partial t} \ln \left(\frac{u-v}{\bar{u}-\bar{v}} \right) + \mathcal{B} \right], \quad \Omega = (\alpha^2 - |u-v|^2)^{1/2}, \quad (2.9)$$

where

$$\mathcal{B} = \frac{1}{3} (\alpha^2 - |u-v|^2) + (|u|^2 + |v|^2), \quad (2.10)$$

and α is a real parameter. Although this is not a completely transparent choice, it is based on knowledge of the NLS Bäcklund transformation and experience with the sine-Gordon equation. It will become clear that (2.9) has nice properties. With the choice (2.9, 2.10) the defect conditions (2.4) at $x = 0$ become:

$$\begin{aligned} u_x &= -\frac{1}{2} \left[\frac{i(u_t - v_t)}{\Omega} - (u+v)\Omega + \frac{(u-v)(|u|^2 + |v|^2)}{\Omega} \right] \\ v_x &= -\frac{1}{2} \left[\frac{i(u_t - v_t)}{\Omega} + (u+v)\Omega + \frac{(u-v)(|u|^2 + |v|^2)}{\Omega} \right], \end{aligned} \quad (2.11)$$

and the time derivative of the momentum simplifies to:

$$P_t = i \frac{\partial}{\partial t} (\bar{u}v - \bar{v}u)_{x=0}, \quad (2.12)$$

and thus the quite attractive combination

$$P - \mathbf{i}(\bar{u}v - \bar{v}u)_{x=0} \quad (2.13)$$

is conserved. The requirement that the time-derivative of the momentum should turn out to be expressible as a functional of the fields evaluated at the defect is actually very strong, and finding an expression which satisfies (2.8) severely limits the choice of defect condition.

On the other hand, as mentioned above, the total energy should be conserved whatever the defect condition is, provided it does not violate time translation invariance. Indeed, the total energy satisfies

$$E_t = \frac{\partial}{\partial t}(\Omega\mathcal{B})_{x=0}, \quad (2.14)$$

implying that $E - (\Omega\mathcal{B})_{x=0}$ is conserved. Notice that the discontinuity $[v - u]_{x=0}$ at the jump-defect should not be too severe. In fact

$$|u - v|_{x=0}^2 \leq \alpha^2, \quad (2.15)$$

otherwise the energy would fail to be real and simply lose its meaning. In the limit $\alpha \rightarrow 0$, the discontinuity disappears. In this sense, the parameter α controls the height of the jump.

There is also a ‘probability’, or ‘number’, density $\mathcal{N} = \bar{u}u$ which satisfies

$$\mathcal{N}_t = \mathbf{i}(\bar{u}_x u - \bar{u} u_x)_x. \quad (2.16)$$

Before checking that the NLS equation remains integrable even after adding a jump-defect it is also worth examining this charge in some detail. On the whole line this ‘number’ is certainly conserved (assuming suitably decaying u at $x = \pm\infty$, or periodic boundary conditions) and it is a consequence (via Noether’s theorem) of the continuous $U(1)$ symmetry of the action under the constant change of phase

$$u \rightarrow e^{\mathbf{i}\Lambda} u \quad \Lambda_x = 0 = \Lambda_t. \quad (2.17)$$

When a jump-defect is added, it is natural to define,

$$N = \int_{-\infty}^0 dx \mathcal{N}(u) + \int_0^{\infty} dx \mathcal{N}(v), \quad (2.18)$$

the defect condition (2.11) leads to

$$N_t = \mathbf{i}(\bar{u}_x u - \bar{u} u_x)_{x=0} - \mathbf{i}(\bar{v}_x v - \bar{v} v_x)_{x=0} = \left. \frac{\partial \Omega}{\partial t} \right|_{x=0}. \quad (2.19)$$

It is clear from this that the combination

$$N + \Omega = N(u) + N(v) - \left(\alpha^2 - |u - v|^2 \right)^{1/2} \Big|_{x=0} \quad (2.20)$$

is conserved. Notice again that the charge makes sense provided the defect is not too severe with $|u - v|_{x=0} < \alpha$.

Notice that adding and subtracting the two defect conditions (2.11) gives the following pair of relations at $x = 0$:

$$\begin{aligned} u_t - v_t &= i(u_x + v_x)\Omega + i(u - v)(|u|^2 + |v|^2) \\ u_x - v_x &= (u + v)\Omega. \end{aligned} \quad (2.21)$$

If a pair of conditions such as these held for every x , and not merely at $x = 0$, (2.21) would be recognised as a Bäcklund transformation: differentiating the second equation with respect to x , using the second equation to substitute for $u_x - v_x$ wherever it appears, and adding i times the first equation gives

$$iu_t + u_{xx} + 2u|u|^2 = iv_t + v_{xx} + 2v|v|^2. \quad (2.22)$$

On the other hand, differentiating the first with respect to x and the second with respect to t and subtracting leads to

$$iu_t + u_{xx} + 2u|u|^2 = -(iv_t + v_{xx} + 2v|v|^2). \quad (2.23)$$

Hence, combining these manipulations, both u and v satisfy the NLS equation. In fact, this is exactly the Bäcklund transformation for NLS given by Lamb [17] (see also [18], [19]).

3. Argument for integrability

In order to verify that introducing a jump-defect does not destroy integrability it is necessary either to investigate additional conserved charges or, more efficiently, to examine a suitably modified Lax pair. The latter approach is generally superior since the Lax pair provides a generating function (as a Laurent series in the spectral parameter) for an infinite set of independent conserved quantities.

A Lax pair for the NLS has been provided in [23] (see also [20]). For the above choice of conventions, a satisfactory Lax pair is:

$$\begin{aligned} L(u) &= i(\bar{u}\sigma_+ + u\sigma_-) + \lambda\sigma_3 \\ M(u) &= i(2\bar{u}u + \lambda^2)\sigma_3 + (\bar{u}_x - \lambda\bar{u})\sigma_+ - (u_x + \lambda u)\sigma_-, \end{aligned} \quad (3.1)$$

where

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \sigma_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (3.2)$$

with the property

$$\partial_t L - \partial_x M + [M, L] = 0 \quad \Longleftrightarrow \quad iu_t + u_{xx} + 2u|u|^2 = 0. \quad (3.3)$$

Moreover, (3.3) holds independently of the choice of spectral parameter λ .

Following the ideas introduced in [8] and [22], modified Lax pairs can be devised which will build in automatically both the bulk equations and the defect conditions. In designing these it is natural to introduce two extra points $a < 0$ and $b > 0$ which are

the endpoints of two regions overlapping the defect, one on the left R^- , $-\infty < x < b$, and one on the right R^+ , $a < x < \infty$. Then, a suitable pair can be defined as follows:

$$\begin{aligned}\hat{L}^- &= L(u) \theta(a - x) \\ \hat{M}^- &= M(u) + \theta(x - a) \left\{ \left[u_x + \frac{i}{2\Omega}(u_t - v_t) - \frac{\Omega}{2}(u + v) + \frac{(u - v)}{2\Omega}(|u|^2 + |v|^2) \right] \sigma_- \right. \\ &\quad \left. - \left[\bar{u}_x - \frac{i}{2\Omega}(\bar{u}_t - \bar{v}_t) - \frac{\Omega}{2}(\bar{u} + \bar{v}) + \frac{(\bar{u} - \bar{v})}{2\Omega}(|u|^2 + |v|^2) \right] \sigma_+ \right\},\end{aligned}\quad (3.4)$$

and

$$\begin{aligned}\hat{L}^+ &= L(v) \theta(x - b) \\ \hat{M}^+ &= M(v) + \theta(b - x) \left\{ \left[v_x + \frac{i}{2\Omega}(u_t - v_t) + \frac{\Omega}{2}(u + v) + \frac{(u - v)}{2\Omega}(|u|^2 + |v|^2) \right] \sigma_- \right. \\ &\quad \left. - \left[\bar{v}_x - \frac{i}{2\Omega}(\bar{u}_t - \bar{v}_t) + \frac{\Omega}{2}(\bar{u} + \bar{v}) + \frac{(\bar{u} - \bar{v})}{2\Omega}(|u|^2 + |v|^2) \right] \sigma_+ \right\}.\end{aligned}\quad (3.5)$$

These modified Lax pairs provide the equations of motion together with all the defect relations as a consequence of zero curvature conditions of the type (3.3). In the overlapping interval $a < x < b$, the two matrices \hat{M}^+ and \hat{M}^- must be x -independent in order to maintain the zero curvature condition, though not necessarily equal. Rather, they must be related by the following ‘gauge’ transformation (see [22] for more details),

$$\partial_t \mathcal{K} = \mathcal{K} \hat{M}^+(b, t) - \hat{M}^-(a, t) \mathcal{K}. \quad (3.6)$$

Note that since \hat{M}^\pm are x -independent this implies that the field u and v are also x -independent in the overlapping interval. It can be verified that the following choice for \mathcal{K}

$$\mathcal{K} = I + \frac{1}{\lambda} \begin{pmatrix} \Omega & -i(\bar{u} - \bar{v}) \\ i(u - v) & -\Omega \end{pmatrix}, \quad (3.7)$$

where I is the 2×2 identity matrix, works perfectly, in the sense that (3.6) is identically satisfied. In the limit $\alpha \rightarrow 0$ the discontinuity tends to zero and $\mathcal{K} \rightarrow I$, as one would expect.

If one was to consider instead the conditions associated with a δ -impurity, namely,

$$u_x = v_x - \sigma v, \quad v_x = u_x + \sigma u, \quad x = 0, \quad (3.8)$$

a similar construction would lead to the conclusion that there was no suitable matrix \mathcal{K} and hence no Lax pair for this type of defect. This suggests the δ -impurity is not integrable, a fact apparently consistent with numerical studies [5], [6]. One could envisage more general linear defect conditions, but the conclusion remains the same.

Once \mathcal{K} is given, and assuming suitably decaying fields at $\pm\infty$, or periodic boundary conditions, conserved quantities will be generated by

$$\mathcal{Q}(\lambda) = \text{Tr} \left[P \exp \left(\int_{-\infty}^a dx L(u) \right) \mathcal{K} P \exp \left(\int_b^\infty dx L(v) \right) \right]. \quad (3.9)$$

The final and necessary step would be to demonstrate that the charges generated by (3.9) are independent and in involution. However, the discussion of this topic will be

postponed. In the meantime, following Ablowitz et al. [24], it will be shown how the Lax pair can be used to generate conserved charges using the integrability constraints represented by the zero curvature condition.

The bulk Lax pair for NLS can be written alternatively as follows

$$L = \begin{pmatrix} \lambda/2 & i\bar{u} \\ iu & -\lambda/2 \end{pmatrix}, \quad M = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \quad (3.10)$$

where the entries of the matrix M can be obtained from (3.1). Applying the zero curvature condition (3.3) to this Lax pair, the following relations are obtained

$$A_x = iuB - i\bar{u}C, \quad \lambda B = 2i\bar{u}A - B_x + i\bar{u}_t, \quad \lambda C = 2iuA + C_x - iu_t. \quad (3.11)$$

These allow the conserved charges to be calculated as the coefficients in an expansion in $1/\lambda$. To see how the argument proceeds, take a couple of steps in detail. Substituting the last two equations of (3.11) into the first of (3.11) it follows that

$$A_x = -\frac{1}{\lambda} [(\bar{u}u)_t + i(uB + \bar{u}C)_x - i(u_xB + \bar{u}_xC)]. \quad (3.12)$$

Repeating the substitution from (3.11) one finds

$$\begin{aligned} A_x = & -\frac{1}{\lambda} [(\bar{u}u)_t + i(uB + \bar{u}C)_x] - \frac{1}{\lambda^2} \left[\frac{1}{2}(\bar{u}u_x - \bar{u}_xu)_t - \frac{1}{2}(\bar{u}u_t - \bar{u}_tu)_x \right] \\ & - \frac{1}{\lambda^2} \left[(2u\bar{u}A + iu_xB - i\bar{u}_xC)_x - (2iu^2\bar{u} + iu_{xx})B + (2i\bar{u}^2u + i\bar{u}_{xx})C \right]. \end{aligned} \quad (3.13)$$

The process can be repeated to obtain further elements of the expansion in higher order in $1/\lambda$. However, (3.13) is enough to illustrate the point. If total x -derivatives integrate to zero (assuming u and its derivatives vanish as $|x| \rightarrow \infty$) then integrating (3.13) over $-\infty < x < \infty$ and concentrating on the first two terms gives

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} dx (\bar{u}u) = 0, \quad \frac{\partial}{\partial t} \int_{-\infty}^{\infty} dx \frac{1}{2}(\bar{u}u_x - \bar{u}_xu) = 0. \quad (3.14)$$

These are the ‘probability or number’ and momentum densities, respectively, used previously.

With a jump-defect the situation is different and it becomes necessary to deal with the modified Lax pairs defined in two regions whose overlap contains the defect. In the matrix form, the Lax pair to use reads as follows

$$\hat{L}^- = \begin{pmatrix} \lambda/2 & i\bar{u} \\ iu & -\lambda/2 \end{pmatrix} \theta(a-x), \quad \hat{M}^- = \begin{pmatrix} \hat{A}^- & \hat{B}^- \\ \hat{C}^- & -\hat{A}^- \end{pmatrix} \theta(b-x) \quad (3.15)$$

$$\hat{L}^+ = \begin{pmatrix} \lambda/2 & i\bar{v} \\ iv & -\lambda/2 \end{pmatrix} \theta(x-b), \quad \hat{M}^+ = \begin{pmatrix} \hat{A}^+ & \hat{B}^+ \\ \hat{C}^+ & -\hat{A}^+ \end{pmatrix} \theta(x-a), \quad (3.16)$$

where the entries of the matrices \hat{M}^- and \hat{M}^+ will not be specified here but they can be obtained from the formulae (3.4) and (3.5). The Lax pairs (3.15) and (3.16) are defined in the regions R^- and R^+ , respectively. Within the overlap the zero curvature condition requires

$$\partial_x \hat{M}^- = \partial_x \hat{M}^+ = 0, \quad (3.17)$$

and therefore $M^{(\pm)}$ is constant throughout the overlap although the two constant values are not necessarily the same. In fact, the ‘gauge’ transformation (3.6) supplies the link between the values of these constants via

$$\hat{M}^+(b, t) = \mathcal{K}^{-1} \partial_t \mathcal{K} + \mathcal{K}^{-1} \hat{M}^-(a, t) \mathcal{K}. \quad (3.18)$$

Next, imagine shrinking the overlap region by allowing its endpoints a, b to approach the defect location at $x = 0$. In these circumstances, the strategy used before, to obtain the conserved charges in the bulk, can be applied. For example, the analogue of (3.12) is

$$\begin{aligned} \hat{A}_x^- + \hat{A}_x^+ = & -\frac{1}{\lambda} \left[(\bar{u}u)_t + (\bar{v}v)_t + i(u\hat{B}_x^- + \bar{u}\hat{C}_x^-)_x + i(u\hat{B}_x^+ + \bar{u}\hat{C}_x^+)_x \right. \\ & \left. - i(u_x\hat{B}_x^- + \bar{u}_x\hat{C}_x^-) - i(u_x\hat{B}_x^+ + \bar{u}_x\hat{C}_x^+) \right], \end{aligned} \quad (3.19)$$

Integrating over $-\infty < x < \infty$ gives

$$\begin{aligned} \left[\hat{A}^- - (\mathcal{K}^{-1} \partial_t \mathcal{K} + \mathcal{K}^{-1} \hat{M}^- \mathcal{K})_{11} \right]_{x=0} = & -\frac{1}{\lambda} \left[\frac{\partial}{\partial t} \int_{-\infty}^0 dx (\bar{u}u) + \frac{\partial}{\partial t} \int_0^{\infty} dx (\bar{v}v) \right] \\ & - \frac{1}{\lambda} \left[i(u\hat{B}_x^- + \bar{u}\hat{C}_x^-) - iu(\mathcal{K}^{-1} \partial_t \mathcal{K} + \mathcal{K}^{-1} \hat{M}^- \mathcal{K})_{12} - i\bar{u}(\mathcal{K}^{-1} \partial_t \mathcal{K} + \mathcal{K}^{-1} \hat{M}^- \mathcal{K})_{21} \right]_{x=0} \\ & + \frac{1}{\lambda} \left[\int_{-\infty}^0 dx i(u_x\hat{B}_x^- + \bar{u}_x\hat{C}_x^-) + \int_0^{\infty} dx i(u_x\hat{B}_x^+ + \bar{u}_x\hat{C}_x^+) \right], \end{aligned} \quad (3.20)$$

where the elements of the matrix (3.18) appear. In turn, these can be obtained using (3.4) and (3.7). Doing so, the first two lines of the integrated expansion (3.20) can be calculated and, schematically, the result is the following

$$\begin{aligned} \frac{1}{\lambda^0} [\cdots]_{x=0} + \frac{1}{\lambda} \left\{ \frac{\partial}{\partial t} \left(\int_{-\infty}^0 dx (\bar{u}u) + \int_0^{\infty} dx (\bar{v}v) \right) + \left[-\frac{\partial \Omega}{\partial t} + \cdots \right]_{x=0} \right\} + \frac{1}{\lambda^2} [\cdots]_{x=0} \\ - \frac{1}{\lambda} \left[\int_{-\infty}^0 dx i(u_x\hat{B}_x^- + \bar{u}_x\hat{C}_x^-) + \int_0^{\infty} dx i(u_x\hat{B}_x^+ + \bar{u}_x\hat{C}_x^+) \right] = 0, \end{aligned} \quad (3.21)$$

where the unwritten parts contain terms in u, v and their spatial derivatives. These terms cancel out after a lengthy calculation. In the first line of (3.21) it is possible to recognize the correction due to the defect, which has to be introduced to make the ‘number’ charge conserved (2.19). The iteration process has been checked up to order $1/\lambda^2$ confirming the correction to the momentum already indicated in (2.12). The details of this verification are straightforward and will be omitted.

It should be clear that these arguments can be repeated for an arbitrary number of jump-defects placed along the x -axis. Each defect is treated locally, introduces its own parameter, and further complicates the generating functional (3.9) for the conserved quantities.

4. A single soliton meeting a defect

It is interesting to investigate what happens as a soliton approaches a jump-defect. With the above normalisation for the nonlinear term in the field equation, a one-soliton

solution is given by

$$u = \frac{2a EF}{1 + E^2}, \quad E = \exp[a(x - 2ct)], \quad F = \exp\left[i\left(cx + (a^2 - c^2)t\right)\right] \quad (4.1)$$

where $a > 0$ and c are free, real parameters. For general values of these parameters, the modulus and phase of the soliton travel with different speeds. A slightly more general solution is obtained by shifting the origin of x , for example, and multiplying u by an arbitrary phase.

When there is a defect, the field v on the other side of it is taken to be given by a similar expression but with different shifts in the modulus and the phase components; in other words, take v to be

$$v = \frac{2a pq EF}{1 + p^2 E^2}. \quad (4.2)$$

In view of the form of the defect conditions it is also reasonable to suppose p is real. Then, the first miracle which should occur concerns the square root in the defect conditions whose argument will have to be a perfect square. For this one already requires the following two relations between p and q :

$$\bar{q}q = 1, \quad \frac{|1 - pq|^4}{(1 - p^2)^2} = \frac{\alpha^2}{a^2}. \quad (4.3)$$

Then, checking the second of equations (2.21) requires

$$pq = \frac{a + ic - \alpha}{a + ic + \alpha}, \quad (4.4)$$

and therefore

$$p^2 = \frac{(a - \alpha)^2 + c^2}{(a + \alpha)^2 + c^2}, \quad q = \frac{a - \alpha + ic}{|a - \alpha + ic|} \frac{a + \alpha - ic}{|a + \alpha + ic|}. \quad (4.5)$$

The latter is quite a nice expression for p since it can never vanish (for $c \neq 0$), and it approaches unity as $c \rightarrow \infty$. Clearly, the soliton cannot be ‘eaten’ by the defect and the faster it travels the less it is affected. Also, p is not sensitive to the sign of c , but changing the sign of c replaces q by \bar{q} . As $|\alpha| \rightarrow \infty$ the parameter $q \rightarrow -1$, indicating that when the defect parameter is sufficiently large the soliton will have inverted its shape when it emerges from the defect (but, since $p \rightarrow 1$, the inverted soliton will not be significantly delayed). As $\alpha \rightarrow 0$, both $p, q \rightarrow 1$ and the effect of the discontinuity disappears, as it should. Notice, the above collection of formulae refer to the positive square root Ω ; changing the sign of the square root will require changing $\alpha \rightarrow -\alpha$ in the above expressions.

Overall, the picture for NLS is quite similar to that discovered previously for sine-Gordon except that in the sine-Gordon case a soliton may be absorbed by the defect, or converted to an anti-soliton [7] according to the choice of defect parameter. These possibilities cannot occur for NLS.

5. The two-soliton solution and a defect

One would expect that a pair of initially widely separated solitons approaching the defect should pass through it independently, each experiencing the jump-defect as if the other soliton were not there. Indeed this is the case although to verify it explicitly is a formidable calculation requiring the use of Mathematica or Maple. In this section the principal steps in this verification will be outlined.

The multi-soliton solution has been given in closed form by Zakharov and Shabat [23] (see also [20]) in terms of the scattering data of the linear problem associated with the NLS. This data is a set of complex constants $\lambda_j = c_j + i a_j$ $a_j > 0$ and functions $\Gamma_j(x, t) = \gamma_j \exp i(\lambda_j x - \lambda_j^2 t)$ where γ_j provides the initial position and phase of the soliton labelled j . The two-soliton solution is then described as follows

$$\frac{u}{2} = \frac{a_1 \Lambda_1 \Gamma_1 + a_2 \Lambda_2 \Gamma_2 + a_1 \bar{\Lambda}_1 \Gamma_1 |\Gamma_2|^2 + a_2 \bar{\Lambda}_2 \Gamma_2 |\Gamma_1|^2}{1 + |\Gamma_1|^2 |\Gamma_2|^2 + |\Lambda_1|^2 |\Gamma_1|^2 + |\Lambda_2|^2 |\Gamma_2|^2 - 4 a_1 a_2 (\Gamma_1 \bar{\Gamma}_2 + \bar{\Gamma}_1 \Gamma_2) / |\lambda_1 - \lambda_2|^2} \quad (5.1)$$

where

$$\Lambda_2 = \frac{(\lambda_2 - \bar{\lambda}_1)}{(\lambda_2 - \lambda_1)}, \quad \Lambda_1 = \frac{(\lambda_1 - \bar{\lambda}_2)}{(\lambda_1 - \lambda_2)}. \quad (5.2)$$

Setting $c = (c_1 - c_2)$, a solution of this type on the left of the defect can be written conveniently as:

$$\frac{u}{2} = \frac{a_1 E_1 F_1 [\Delta_+ (1 + E_2^2) - 2i c a_2 (1 - E_2^2)] - a_2 E_2 F_2 [\Delta_- (1 + E_1^2) - 2i c a_1 (1 - E_1^2)]}{(c^2 + \delta_-^2)(1 + E_1^2 E_2^2) + (c^2 + \delta_+^2)(E_1^2 + E_2^2) - 4 a_1 a_2 E_1 E_2 (F_1^2 + F_2^2) / F_1 F_2} \quad (5.3)$$

where $\delta_+ = (a_1 + a_2)$, $\delta_- = (a_1 - a_2)$, $\Delta_{\pm} = \delta_+ \delta_- \pm c^2$ and

$$E_j = \exp [a_j (x - c_j t) - a_j x_{0j}], \quad F_j = \exp [i(c_j x + (a_j^2 - c_j^2)t + \phi_{0j})] \quad j = 1, 2. \quad (5.4)$$

Note that expression (5.3) contains the one-soliton solution (4.1) on setting, for instance, $a_2 = c_2 = 0$.

To the right of the defect the expression (5.3) will be modified as follows

$$\frac{v}{2} = \frac{a_1 G_1 [\Delta_+ (1 + H_2^2) - 2i c a_2 (1 - H_2^2)] - a_2 G_2 [\Delta_- (1 + H_1^2) - 2i c a_1 (1 - H_1^2)]}{(c^2 + \delta_-^2)(1 + H_1^2 H_2^2) + (c^2 + \delta_+^2)(H_1^2 + H_2^2) - 4 a_1 a_2 (G_1^2 H_2^2 + G_2^2 H_1^2) / (G_1 G_2)} \quad (5.5)$$

where $G_j = z_j E_j F_j$, $H_j^2 = w_j E_j^2$ and $w_j = p_j^2$, $z_j = p_j q_j$ with $j = 1, 2$ and $|q_1| = |q_2| = 1$. The constants p_j , q_j represent the delays in position and phase, respectively, for the two solitons residing inside the solution v , by analogy with the notation used for the one-soliton solution on each side of the defect.

The first step in exploring the consequences of the defect conditions is to find the circumstances under which the argument of the square root $\Omega = (\alpha^2 - |u - v|^2)^{1/2}$ is a perfect square. The most general polynomial whose square can match the argument of the square root must have the following form,

$$\begin{aligned} \Omega = & (a + b_1 E_1^2 + b_2 E_2^2 + d_1 E_1^4 + d_2 E_2^4 + e_1 E_1^4 E_2^2 + e_2 E_1^2 E_2^4 \\ & + g_1 E_1^3 E_2 + g_2 E_1 E_2^3 + h E_1 E_2 + r E_1^3 E_2^3 + s E_1^4 E_2^4 + f E_1^2 E_2^2). \end{aligned} \quad (5.6)$$

The coefficients of Ω^2 are vastly overdetermined and it is expected that just a few relationships among the parameters will suffice. Indeed, this turns out to be the case, and the following constraints on the constants w_j, z_j are all that is required,

$$[a_j(z_j - 1)(w_j - z_j)]^2 = [\alpha(1 - w_j)z_j]^2 \quad j = 1, 2 \quad (5.7)$$

$$\begin{aligned} (a_1 + a_2 - i\alpha)(w_1 - z_1)(z_2 - 1) &= 2\alpha(z_1 - w_1z_2) & a_i \neq 0, \quad i = 1, 2 \\ (a_1 - a_2 + i\alpha)(z_1 - 1)(z_2 - 1) &= 2\alpha(z_1 - z_2) & a_i \neq 0, \quad i = 1, 2. \end{aligned} \quad (5.8)$$

Note that, contrary to relations (5.7), the expressions (5.8) are not real and therefore their complex conjugate partners have also to be taken into account

$$\begin{aligned} (a_1 + a_2 + i\alpha)(z_1 - 1)(w_2 - z_2) &= 2\alpha(z_2 - w_2z_1) & a_i \neq 0, \quad i = 1, 2 \\ (a_1 - a_2 - i\alpha)(w_1 - z_1)(w_2 - z_2) &= 2\alpha(w_1z_2 - w_2z_1) & a_i \neq 0, \quad i = 1, 2. \end{aligned} \quad (5.9)$$

Implementing these constraints, the second part of the defect condition (2.21) is satisfied provided the following two relations hold

$$z_j = \frac{a_j - \alpha + i c_j}{a_j + \alpha + i c_j} \quad j = 1, 2 \quad (5.10)$$

and therefore

$$w_j = \frac{(a_j - \alpha)^2 + c_j^2}{(a_j + \alpha)^2 + c_j^2} \quad j = 1, 2. \quad (5.11)$$

Matching the conditions in this way demonstrates that solitons are transmitted through the defect independently of one another. It is expected this property should hold for arbitrary numbers of solitons although there is no general proof of that yet.

In the sine-Gordon model it has been remarked that the delay experienced by a soliton passing a defect is actually the square root of the delay that would be experienced by the same soliton passing another whose rapidity was equal to the defect parameter [13]. Using the following change of variables suggested in [23] (see also [20]) when $c_1 > c_2$,

$$\Gamma_1^+ = \Gamma_1 \Lambda_1 \quad \Gamma_1^- = \frac{\Gamma_1}{\Lambda_1} \quad \Gamma_2^+ = \frac{\Gamma_2}{\Lambda_2} \quad \Gamma_2^- = \Gamma_2 \Lambda_2, \quad (5.12)$$

the two soliton solution (5.1) may be rewritten as follows

$$\frac{u}{2} = \frac{a_1 \Gamma_1^+ + a_2 \Gamma_2^- + a_1 \Gamma_1^- |\Gamma_2^-|^2 + a_2 \Gamma_2^+ |\Gamma_1^+|^2}{1 + |\Gamma_1^+|^2 + |\Gamma_2^-|^2 + |\Gamma_1^+|^2 |\Gamma_2^+|^2 - 8 a_1 a_2 \text{Re}[(\Gamma_1^+ \bar{\Gamma}_2^-)/(\lambda_1 - \bar{\lambda}_2)^2]}. \quad (5.13)$$

Note that the change of variables (5.12) always holds provided $\lambda_1 \neq \lambda_2, \bar{\lambda}_2$. Moreover

$$|\Gamma_1^+|^2 |\Gamma_2^+|^2 = |\Gamma_1^-|^2 |\Gamma_2^-|^2 = |\Gamma_1|^2 |\Gamma_2|^2. \quad (5.14)$$

Functions $\Gamma_j^\pm(x, t)$ are defined as follows

$$\Gamma_j^\pm(x, t) = \gamma_j^\pm \exp[i(\lambda_j x - \lambda_j^2 t)], \quad \gamma_j^\pm = \exp(a_j x_{0j}^\pm + i \phi_{0j}^\pm) \quad (5.15)$$

where now

$$\begin{aligned} \Delta p_0 &= \exp[-a_1(x_{01}^+ - x_{01}^-)] = \exp[a_2(x_{02}^+ - x_{02}^-)], \\ \Delta q_{01} &= \exp[-i(\phi_{01}^+ - \phi_{01}^-)], \quad \Delta q_{02} = \exp[i(\phi_{02}^+ - \phi_{02}^-)] \end{aligned} \quad (5.16)$$

are the two-soliton scattering data. In fact, looking at the limit $t = \pm\infty$ of the two-soliton solution (5.13), for instance along the first soliton trajectory $x - c_1 t = \text{constant}$, it is possible to obtain the following one-soliton solutions ($c_1 > c_2$)

$$u|_{t \rightarrow \infty} = \frac{2 a_1 \Gamma_1^+}{1 + |\Gamma_1^+|^2}, \quad u|_{t \rightarrow -\infty} = \frac{2 a_1 \Gamma_1^-}{1 + |\Gamma_1^-|^2}. \quad (5.17)$$

Using (5.12), explicit expressions for the scattering data Δp_0 , Δq_{01} and Δq_{02} are:

$$\Delta p_0 = \frac{(a_1 - a_2)^2 + c^2}{(a_1 + a_2)^2 + c^2}, \quad (5.18)$$

$$\begin{aligned} (\Delta q_{01})^{-1} &= \exp[i(\phi_{01}^+ - \phi_{01}^-)] = \frac{a_1^2 - a_2^2 + 2ia_2c + c^2}{a_1^2 - a_2^2 - 2ia_2c + c^2}, \\ \Delta q_{02} &= \frac{a_2^2 - a_1^2 + 2ia_1c + c^2}{a_2^2 - a_1^2 - 2ia_1c + c^2}. \end{aligned} \quad (5.19)$$

Note that $(\Delta q_{01})^{-1}$ can be also written as follows

$$(\Delta q_{01})^{-1} = \frac{(a_1^2 - a_2^2)^2 + c^4 + 2c^2a_1^2 - 6c^2a_2^2 + 4ia_2(c^2 + a_1^2 - a_2^2)}{(a_1^2 - a_2^2)^2 + c^4 + 2c^2(a_1^2 + a_2^2)}. \quad (5.20)$$

It is then easy to check that Δp_0 and $(\Delta q_{01})^{-1}$ coincide with the squares of the shifts in the modulus p and the phase q (4.5) experienced by the soliton progressing through a defect, provided $c_2 = 0$ and $a_2 = \alpha$, with α being the defect parameter.

6. Bound states

With a delta function impurity, in a free field situation with a suitable coupling, there will be a bound state. Curiously, this is not the case for the free field limit of the sine-Gordon model with the jump-defect conditions provided by (1.2), nor is it the case for the full sine-Gordon model with a jump-defect. In the sine-Gordon model this type of purely transmitting defect does not permit bound states for any value of the coupling σ . Allowing a jump-defect in the quantum sine-Gordon model does have interesting effects, however, and it appears a defect can be excited, albeit with a finite decay width. This is the quantum analogue of the classical property permitting a jump-defect to swallow a soliton [13].

On the other hand, a jump-defect in the NLS model certainly has bound states associated with it. The first indication of this arises using the linear, potential-free, Schrödinger equations and the linearised versions of the defect conditions (2.21)

$$\begin{aligned} u_x &= -\frac{i}{2\alpha}(u_t - v_t) + \frac{\alpha}{2}(u + v) \\ v_x &= -\frac{i}{2\alpha}(u_t - v_t) - \frac{\alpha}{2}(u + v). \end{aligned} \quad (6.1)$$

It is easy to check these allow the travelling wave solution

$$u = u_0 \exp(-ik^2t + ikx), \quad v = v_0 \exp(-ik^2t + ikx), \quad v_0 = \frac{k + i\alpha}{k - i\alpha} u_0, \quad (6.2)$$

where k is real, from which the existence of bound states associated with the defect may be deduced. If $\alpha > 0$ (< 0), (6.2) indicates a bound state when $k = i\alpha$ ($-i\alpha$) for which there are the square integrable solutions

$$u = 0, \quad v = v_0 \exp(i\alpha^2 t - \alpha x); \quad (u = u_0 \exp(i\alpha^2 t + \alpha x), \quad v = 0), \quad (6.3)$$

respectively.

Interestingly, these bound states have their counterparts in the fully nonlinear version of the model. For example, taking α to be positive, with $a = \alpha$, $c = 0$ in (4.1), the field configurations

$$u = \frac{2\alpha e^{\alpha(x-x_0)} e^{i\alpha^2 t}}{1 + e^{2\alpha(x-x_0)}}, \quad v = 0, \quad (6.4)$$

in the regions $x < 0$ and $x > 0$, respectively, satisfy the jump defect conditions. If the parameter $x_0 = 0$, the quantity Ω actually vanishes at the defect, as does the momentum of course, while $N = \alpha$ and $E = -\alpha^3/3$ are exactly half the values the number and energy would have had for a static breather solution (with $a = \alpha$, $c = 0$) in the bulk. In fact, the latter values for N and E remain correct if $x_0 \neq 0$; in those cases Ω does not vanish but compensates the bulk integrals. Note, this solution is entirely consistent with the expressions discovered before in (4.5), noting that if $c = 0$ then p vanishes precisely when $a = \alpha$.

7. The modified Korteweg-de Vries

One might wonder if the modified KdV (mKdV) equation, or the KdV itself, both of which certainly possesses Bäcklund transformations, also allow discontinuous solutions, or defects, provided suitable conditions are imposed.

In the bulk, a mKdV equation is [21]

$$v_t - 6v^2 v_x - v_{xxx} = 0, \quad (7.1)$$

or, setting $v = p_x$,

$$p_{xt} - 6p_x^2 p_{xx} - p_{xxxx} = 0, \quad (7.2)$$

the latter being suitable for a Lagrangian description with Lagrangian density

$$\mathcal{L} = \frac{1}{2} p_x p_t - \frac{1}{2} (p_x)^4 + \frac{1}{2} (p_{xx})^2. \quad (7.3)$$

Integrating (7.2) once with respect to x , and assuming all derivatives are asymptotically vanishing gives an alternative:

$$p_t - 2p_x^3 - p_{xxx} = 0. \quad (7.4)$$

There is another version of mKdV (obtained by the replacement $v \rightarrow iv$) in which the middle term on the left hand side of (7.1) changes sign. However, it is equation (7.1) which allows real soliton solutions (see below), not the alternative version [24].

Over the whole line, the quantity

$$P = \frac{1}{2} \int_{-\infty}^{\infty} dx p_x^2, \quad (7.5)$$

is conserved (with the usual assumptions at $\pm\infty$) because

$$\left(\frac{p_x^2}{2}\right)_t = \left(p_t p_x - \frac{1}{2}p_x^4 - \frac{p_{xx}^2}{2}\right)_x, \quad (7.6)$$

where the last expression made use of the alternative equation (7.4).

Next, suppose there is a ‘defect’ at $x = 0$, with fields p, q on either side of it. The quantity P defined by

$$P = \frac{1}{2} \int_{-\infty}^0 dx p_x^2 + \frac{1}{2} \int_0^{\infty} dx q_x^2 \quad (7.7)$$

will not be conserved but, as a consequence of (7.6), its time derivative will be related to a boundary term as follows,

$$P_t = \left(p_t p_x - \frac{p_x^4}{2} - \frac{p_{xx}^2}{2}\right)_{x=0} - \left(q_t q_x - \frac{q_x^4}{2} - \frac{q_{xx}^2}{2}\right)_{x=0}. \quad (7.8)$$

The question is how to write the latter as a time derivative of the fields and their derivatives evaluated at $x = 0$.

To see where the defect conditions are coming from, it is necessary to return and consider the complete action

$$\mathcal{A} = \int dt \left\{ \int_{-\infty}^0 dx \mathcal{L}(p) + \mathbf{B} + \int_0^{\infty} dx \mathcal{L}(q) \right\}, \quad (7.9)$$

and its variation with respect to p or q . Thus, for example, varying p gives,

$$\delta A = \int dt \left\{ \int_{-\infty}^0 dx \left(\frac{1}{2} \delta p_t p_x + \frac{1}{2} \delta p_x p_t - 2 \delta p_x p_x^3 + \delta p_{xx} p_{xx} \right) \right. \quad (7.10)$$

$$\left. + \left(\delta p_t \frac{\partial \mathbf{B}}{\partial p_t} + \delta p_x \frac{\partial \mathbf{B}}{\partial p_x} + \delta p_{xx} \frac{\partial \mathbf{B}}{\partial p_{xx}} \right)_{x=0} \right\}, \quad (7.11)$$

which, on integrating the first term by parts with respect to t and x , keeping the boundary terms in x , and setting the variation to zero, leads to the field equation for p together with

$$0 = \left[\delta p \left(\frac{1}{2} p_t - 2 p_x^3 - p_{xxx} + \frac{\partial \mathbf{B}}{\partial p} - \frac{\partial}{\partial t} \frac{\partial \mathbf{B}}{\partial p_t} \right) + \delta p_x \left(p_{xx} + \frac{\partial \mathbf{B}}{\partial p_x} \right) \right]_{x=0}. \quad (7.12)$$

At the defect, there is no necessity for δp and δp_x to be related but (7.4) can be used (in a limiting sense) to eliminate p_{xxx} . Hence, the defect conditions on p should read

$$0 = -\frac{1}{2} p_t + \frac{\partial \mathbf{B}}{\partial p} - \frac{\partial}{\partial t} \frac{\partial \mathbf{B}}{\partial p_t}, \quad 0 = p_{xx} + \frac{\partial \mathbf{B}}{\partial p_x}. \quad (7.13)$$

Similarly, the relations to be satisfied by q at the defect are:

$$0 = \frac{1}{2} q_t + \frac{\partial \mathbf{B}}{\partial q} - \frac{\partial}{\partial t} \frac{\partial \mathbf{B}}{\partial q_t}, \quad 0 = -q_{xx} + \frac{\partial \mathbf{B}}{\partial q_x}. \quad (7.14)$$

The next step is to find a suitable defect term \mathbf{B} . However, this is not quite straightforward and before doing so it is worth making a remark.

It is tempting to assume the defect term depends only on p, q and their first derivatives (in fact this was tacitly assumed above). However, there is no reason in principle why this term should not depend on p_{xx} and q_{xx} since the field equations

cannot at the defect relate these derivatives to anything else (but note, the same would not be true of p_{xxx} , or q_{xxx}). Moreover, since the Lagrangian (7.3) depends upon both first and second derivatives, it should not be surprising that defect conditions should also involve higher spatial derivatives. If \mathbf{B} does depend on these derivatives then there will be two more defect conditions coming from varying the boundary term alone, unbalanced by anything in the bulk; they are

$$\frac{\partial \mathbf{B}}{\partial p_{xx}} = 0, \quad \frac{\partial \mathbf{B}}{\partial q_{xx}} = 0. \quad (7.15)$$

On the other hand, a Bäcklund transformation for the mKdV equation was found long ago [17, 25] and can be written in the following symmetrical manner:

$$\begin{aligned} p_x + q_x &= \alpha \sin(p - q), \\ p_t + q_t &= \alpha (p_{xx} - q_{xx}) \cos(p - q) + \alpha (p_x^2 + q_x^2) \sin(p - q) \\ &= \alpha (p_{xx} - q_{xx}) \cos(p - q) - 2\alpha p_x q_x \sin(p - q) + \alpha^3 \sin^3(p - q). \end{aligned} \quad (7.16)$$

The parameter α is arbitrary and the pair of equations is symmetrical under interchanging p and q , and simultaneously making the replacement $\alpha \rightarrow -\alpha$. In the bulk, the derivative of the first of the pair (7.16) is

$$p_{xx} + q_{xx} = \alpha (p_x - q_x) \cos(p - q), \quad (7.17)$$

which will provide the extra equation coming from (7.15) (see below). Note that, under the circumstances being explored here, the x -derivatives are frozen, therefore (7.17) cannot be a consequence of the first of conditions (7.16).

Using (7.16) and (7.17) the expression (7.8) can be simplified to

$$P_t = \frac{d}{dt} [-\alpha \cos(p - q)]_{x=0}, \quad (7.18)$$

and therefore,

$$P + \alpha [\cos(p - q) - 1]_{x=0} \quad (7.19)$$

is conserved. The constant has been chosen to ensure the momentum stored at the defect is zero when there is no discontinuity. It is worth noting that because of the discontinuity the charge that in the bulk is obtained from the density $\mathcal{N} = p_x$ is not conserved. The same could be said for the KdV model discussed in section 9.

As previously, it is useful to put

$$\mathbf{B} = \frac{1}{4}(qp_t - pq_t) - \mathcal{B} \quad (7.20)$$

and the defect conditions may be rewritten

$$(p_t + q_t) = -2\frac{\partial \mathcal{B}}{\partial p} = 2\frac{\partial \mathcal{B}}{\partial q}; \quad p_{xx} = \frac{\partial \mathcal{B}}{\partial p_x}, \quad q_{xx} = -\frac{\partial \mathcal{B}}{\partial q_x}; \quad 0 = \frac{\partial \mathcal{B}}{\partial p_{xx}} = \frac{\partial \mathcal{B}}{\partial q_{xx}}. \quad (7.21)$$

Putting everything together, a suitable choice of \mathcal{B} appears to be to take

$$\begin{aligned} \mathcal{B} &= \frac{1}{2} (p_{xx} - q_{xx}) (p_x + q_x - \alpha \sin(p - q)) \\ &\quad + \frac{\alpha}{6} \cos(p - q) [p_x^2 + q_x^2 - 4p_x q_x + \alpha (p_x + q_x) \sin(p - q) + \alpha^2]. \end{aligned} \quad (7.22)$$

Using (7.22) the defect conditions are exactly equivalent to what would be the Bäcklund transformation in the bulk together with (7.17) were the conditions not frozen at $x = 0$. Finally, it is worth remarking that the mKdV defect potential is not a simple function of the two fields u and v and the parameter α but rather of the ‘potentials’ p and q .

8. The mKdV soliton passing through a defect

The single soliton for the mKdV equation can be conveniently written in terms of p in the form

$$e^{ip} = \frac{1 + iE}{1 - iE}, \quad E = \exp[a(x - x_0 + a^2t)], \quad (8.1)$$

where a and $\exp(x_0)$ are both real parameters. With the conventions adopted in the last section, the soliton moves from right to left along the x -axis. Since the derivative of the above expression is either always positive ($E > 0$) or always negative ($E < 0$), there is also a matching anti-soliton obtained by replacing $E \rightarrow -E$ in (8.1). This is easily seen, on noting

$$p_x = \frac{2aE}{1 + E^2}. \quad (8.2)$$

If there is a defect, the soliton on the other side of it will have the form

$$e^{iq} = \frac{1 + izE}{1 - izE}, \quad E = \exp[a(x - x_0 + a^2t)], \quad (8.3)$$

where z is a parameter to be determined by the defect condition.

The first of the defect conditions readily reveals that

$$z = \frac{\alpha - a}{\alpha + a}. \quad (8.4)$$

This appears to suggest that a slow soliton is not affected much whereas a fast soliton has $z < 0$. This means that a fast soliton flips to an anti-soliton ($E \rightarrow -E$ in the expression (8.1)). When $a = \alpha$, the soliton is eaten. This behaviour is very similar to that of a soliton meeting a defect in the sine-Gordon model where all these effects are similarly apparent. Despite the non-locality, there appears to be nothing particularly pathological about this case. As a final remark in this section, it is clear the effect of the defect disappears as $\alpha \rightarrow \infty$.

9. The Korteweg-de Vries equation

It is not difficult to repeat these steps for the KdV equation. However, there does seem to be some curiously different behaviour that will become apparent as this section proceeds.

In the bulk, the KdV equation is [21]

$$u_t - 6uu_x + u_{xxx} = 0, \quad (9.1)$$

or, setting $u = p_x$,

$$p_{xt} - 6p_x p_{xx} + p_{xxx} = 0, \quad (9.2)$$

the latter being suitable for a Lagrangian description with Lagrangian density

$$\mathcal{L} = \frac{1}{2}p_x p_t - (p_x)^3 - \frac{1}{2}(p_{xx})^2. \quad (9.3)$$

Integrating (9.2) once with respect to x , and assuming all derivatives are asymptotically vanishing gives an alternative:

$$p_t - 3p_x^2 + p_{xxx} = 0. \quad (9.4)$$

Over the whole line, the quantity

$$P = \frac{1}{2} \int_{-\infty}^{\infty} dx (p_x)^2, \quad (9.5)$$

is conserved (with the usual assumptions at $\pm\infty$) because

$$\left(\frac{p_x^2}{2}\right)_t = \left(2p_x^3 - p_x p_{xxx} + \frac{p_{xx}^2}{2}\right)_x = \left(p_t p_x - p_x^3 + \frac{p_{xx}^2}{2}\right)_x, \quad (9.6)$$

where the last expression made use of the alternative equation (9.4).

Next, suppose there is a ‘defect’ at $x = 0$, with fields p, q on either side of it. The quantity P defined by

$$P = \frac{1}{2} \int_{-\infty}^0 dx p_x^2 + \frac{1}{2} \int_0^{\infty} dx q_x^2 \quad (9.7)$$

is not conserved but, as a consequence of (9.6), its time derivative will be related to a boundary term as follows,

$$P_t = \left(p_t p_x - p_x^3 + \frac{p_{xx}^2}{2}\right)_{x=0} - \left(q_t q_x - q_x^3 + \frac{q_{xx}^2}{2}\right)_{x=0}. \quad (9.8)$$

The question is how to write the latter as a time derivative of the fields or their derivatives evaluated at $x = 0$.

The right hand side of (9.8) can be organised a little differently (dropping the explicit reference to the point $x = 0$, which is to be understood from now on) to,

$$-(p_x - q_x)(p_x^2 + p_x q_x + q_x^2) + \frac{1}{2}(p_{xx} - q_{xx})(p_{xx} + q_{xx}) + p_t p_x - q_t q_x, \quad (9.9)$$

and then simplified by setting

$$\begin{aligned} p_{xx} + q_{xx} &= (p - q)(p_x - q_x) \\ p_x^2 + p_x q_x + q_x^2 &= \frac{1}{2}[p_t + q_t + (p - q)(p_{xx} - q_{xx})], \end{aligned} \quad (9.10)$$

to get

$$\frac{1}{2}(p_t - q_t)(p_x + q_x). \quad (9.11)$$

Finally, setting in addition

$$p_x + q_x = 2\alpha + \frac{1}{2}(p - q)^2, \quad (9.12)$$

one finds

$$P_t = \frac{d}{dt} \left[\alpha(p - q) + \frac{1}{12}(p - q)^3 \right]_{x=0}, \quad (9.13)$$

where the parameter α is arbitrary. The first of the conditions (9.10) would follow from (9.12) in the bulk but, here, as already mentioned before, the x -derivatives are frozen. The second of equations (9.10) together with (9.12) provides a Bäcklund transformation for KdV in the bulk; at least in the sense that if p satisfies (9.2) so does q (or vice-versa), for any choice of α . This is the form of Bäcklund transformation constructed by Wahlquist and Estabrook [26] for generating multi-soliton solutions to the KdV equation.

This time using the Lagrangian together with a defect contribution (assuming the latter depends on p and q and their derivatives p_x , q_x and p_{xx} , q_{xx}), leads naturally to the defect conditions:

$$0 = -\frac{1}{2}p_t + \frac{\partial \mathbf{B}}{\partial p} - \frac{\partial}{\partial t} \frac{\partial \mathbf{B}}{\partial p_t}, \quad 0 = -p_{xx} + \frac{\partial \mathbf{B}}{\partial p_x}, \quad 0 = \frac{\partial \mathbf{B}}{\partial p_{xx}}. \quad (9.14)$$

Similarly, the relations to be satisfied by q at the defect are:

$$0 = \frac{1}{2}q_t + \frac{\partial \mathbf{B}}{\partial q} - \frac{\partial}{\partial t} \frac{\partial \mathbf{B}}{\partial q_t}, \quad 0 = q_{xx} + \frac{\partial \mathbf{B}}{\partial q_x}, \quad 0 = \frac{\partial \mathbf{B}}{\partial q_{xx}}. \quad (9.15)$$

A suitable choice for the jump-defect potential might be the following

$$\begin{aligned} \mathbf{B} = & \frac{1}{4}(qp_t - pq_t) + (p_x^2 + p_x q_x + q_x^2)(p - q) + \frac{1}{2}(p_{xx} - q_{xx}) \left[p_x + q_x - 2\alpha - \frac{1}{2}(p - q)^2 \right] \\ & - 3(p_x + q_x)(p - q) \left(\alpha + \frac{1}{4}(p - q)^2 \right) + 6\alpha^2(p - q) + 2\alpha(p - q)^3 + \frac{9}{40}(p - q)^5. \end{aligned} \quad (9.16)$$

This looks quite complicated but it does not seem easy to find anything simpler that would be able to provide the relations needed. In this respect, both the last equations in (9.14), (9.15) give precisely the equation (9.12), while two of the other conditions, still from (9.14), (9.15), give

$$p_{xx} = -\frac{1}{2}(p_{xx} - q_{xx}) + (2p_x + q_x)(p - q) - 3\alpha(p - q) - \frac{3}{4}(p - q)^3, \quad (9.17)$$

$$q_{xx} = -\frac{1}{2}(p_{xx} - q_{xx}) - (p_x + 2q_x)(p - q) + 3\alpha(p - q) + \frac{3}{4}(p - q)^3. \quad (9.18)$$

Adding these gives

$$p_{xx} + q_{xx} = (p_x - q_x)(p - q), \quad (9.19)$$

which, in the bulk, is just the derivative of (9.12); subtracting them gives (9.12) again (the second derivatives exactly cancelling out).

Finally, the other pair of equations from (9.14), (9.15) give effectively the same condition, namely

$$\frac{1}{2}(p_t + q_t) = -\frac{1}{2}(p_{xx} - q_{xx})(p - q) + (p_x^2 + p_x q_x + q_x^2). \quad (9.20)$$

This is not quite straightforward to see and makes use of (9.12) again. Indeed, the coefficients of the polynomial part of \mathbf{B} were chosen to ensure this worked out.

With this particular choice of \mathbf{B} , the ‘momentum’ is compensated in precisely the manner envisaged before (in (9.13)), and

$$P - \left[\alpha(p - q) + \frac{1}{12}(p - q)^3 \right]_{x=0} \quad (9.21)$$

is conserved. Moreover, the parameter α is completely free.

The next bulk conserved quantity, the ‘energy’, has a density

$$\mathcal{E} = \left[(p_x)^3 + \frac{1}{2}(p_{xx})^2 \right], \quad (9.22)$$

It is not conserved but the defect contributes exactly

$$E_t = \left(-\frac{d\mathcal{B}}{dt} \right)_{x=0}, \quad (9.23)$$

provided the defect potential (9.16) is written

$$\mathbf{B} = \frac{1}{4}(qp_t - pq_t) - \mathcal{B}. \quad (9.24)$$

Hence, using the boundary conditions

$$E + \mathcal{B} = E - \left[(p_x^2 + p_x q_x + q_x^2)(p - q) + (p - q)^3 \left(\alpha + \frac{3}{20}(p - q)^2 \right) \right]_{x=0} \quad (9.25)$$

is conserved.

The situation is fairly similar to what was found for mKdV. It is curious that neither of the compensating terms can be rewritten in terms of the the original KdV variables u on the left, or v on the right ($u = p_x$, $v = q_x$). The expressions for energy and momentum reveal that there is nothing particularly special about a vanishing parameter α .

In general, provided the defect potential \mathbf{B} is chosen as it is in (9.24), the momentum is conserved provided

$$\left(\frac{\partial \mathcal{B}}{\partial p_x} \right)^2 - \left(\frac{\partial \mathcal{B}}{\partial q_x} \right)^2 + (p_x - q_x) \left[2 \frac{\partial \mathcal{B}}{\partial p} + (p_x^2 + q_x^2)(p_x + q_x) \right] = 0, \quad (9.26)$$

for mKdV and

$$\left(\frac{\partial \mathcal{B}}{\partial p_x} \right)^2 - \left(\frac{\partial \mathcal{B}}{\partial q_x} \right)^2 - (p_x - q_x) \left[2 \frac{\partial \mathcal{B}}{\partial p} + 2(p_x^2 + p_x q_x + q_x^2) \right] = 0, \quad (9.27)$$

for KdV, respectively, where

$$\left(\frac{\partial \mathcal{B}}{\partial p_x} \right)^2 - \left(\frac{\partial \mathcal{B}}{\partial q_x} \right)^2 = (p_{xx} + q_{xx})(p_{xx} - q_{xx}), \quad 2 \frac{\partial \mathcal{B}}{\partial p} = -(p_t + q_t), \quad (9.28)$$

together with

$$(p_t - q_t)(p_x + q_x) = 2 \frac{d\mathcal{F}}{dt}, \quad (9.29)$$

for either model, where \mathcal{F} is a function to be determined.

10. KdV solitons and the defect

A single soliton can be described conveniently by choosing

$$p = p_0 - \frac{2aE}{1+E}, \quad E = \exp[a(x - x_0 - a^2t)] \equiv \rho \exp[a(x - a^2t)], \quad (10.1)$$

where p_0, a, x_0 are constants. In terms of the original variable u the solution has the well-known characteristic ‘bell-shaped’ form

$$u = p_x = -\frac{2a^2E}{(1+E)^2}. \quad (10.2)$$

The parameter a is real and the soliton is independent of the sign of a (since p_x is the same whatever the sign). Inevitably, the soliton moves to the right.

Then, it is not difficult to check that picking a similar form for q

$$q = q_0 - \frac{2a\sigma E}{1+\sigma E}, \quad E = \exp[a(x - a^2t)], \quad (10.3)$$

and redefining the defect parameter $\alpha = -\beta^2/4$, equation (9.12) implies

$$(p_0 - q_0)^2 = \beta^2, \quad \sigma = \left(\frac{|\beta| - a}{|\beta| + a} \right) \rho. \quad (10.4)$$

At least that is the case assuming the positive square root is taken for $p_0 - q_0$. The other equation (the second of (9.10)) is just an identity using only the fact

$$p_0 - q_0 = a \left(\frac{\rho + \sigma}{\rho - \sigma} \right), \quad (10.5)$$

as can be checked easily using an algebraic computing package. Finally, the first of conditions (9.10) is also an identity and can be checked directly. The relation (10.5) does not depend on taking square roots.

This implies a soliton encountering the defect, provided it is not travelling too quickly, will be delayed. When a approaches β , σ tends to zero and the soliton will have been ‘eaten’ (since $q = q_0$). There is a mystery if the soliton is moving too quickly because the solution on the right of the defect develops a singularity. This is puzzling for another reason. If the defect parameter is taken to be very small then only a very slow soliton will remain non-singular, meaning that in the limit as the parameter goes to zero one cannot recover a single bulk soliton because the speed of the non-singular soliton would also have to approach zero. Similarly, in the limit of large defect parameter $(p_0 - q_0)^2 \rightarrow \infty$, implying that the effect of the defect on the soliton does not entirely disappear, although the fields to either side of it will be the same (because $p_x \rightarrow q_x$). In these respects, the KdV defect behaves curiously differently from those encountered elsewhere.

One might wonder whether having a single soliton on the left of the defect and a double soliton on the right might provide some hints towards solving the mystery. To explore this, consider the following single one-soliton on the left and two-soliton

solution on the right of the defect, respectively

$$\begin{aligned} p &= p_0 - \frac{2 a_2 \rho_2 E_2}{1 + \rho_2 E_2}, \\ q &= q_0 + \frac{(a_1^2 - a_2^2)(1 + \rho_1 E_1 + \sigma \rho_2 E_2 + \sigma \rho_1 \rho_2 E_1 E_2)}{(a_1 - a_2)(1 - \sigma \rho_1 \rho_2 E_1 E_2) - (a_1 + a_2)(\rho_1 E_1 - \sigma \rho_2 E_2)}, \end{aligned} \quad (10.6)$$

with

$$\rho_j E_j = \exp \left[a_j (x - x_{0j} - a_j^2 t) \right] \quad j = 1, 2. \quad (10.7)$$

As pointed out in [26], a two-soliton solution is non-singular provided one of the two component solitons is actually singular. For instance, in (10.6) suppose $a_2 > a_1$, then for q to be regular $\rho_1 > 0$ and $\sigma \rho_2 < 0$ and consequently the faster soliton is singular. Checking the Bäcklund equation (9.12) for the solution (10.6) reveals the following

$$a_1^2 = \beta^2, \quad (p_0 - q_0)^2 = \beta^2, \quad \sigma = 1; \quad (10.8)$$

therefore, provided the incoming soliton is regular ($\rho_2 > 0$), the resulting two-soliton solution is singular. One could think that the ‘fast’ soliton is trapped by the defect since it does not have enough energy to escape. On the other hand, if the one-soliton solution on the left is singular ($\rho_2 > 0$) then the resulting two-soliton solution would be regular (still with $a_2 > a_1$). The singularity of the incoming soliton can be kept to the right of the defect by a suitable choice of the constant x_{02} , in fact, simply taking $x_{02} > 0$ will suffice. Such a singularity remains on the right of the defect as $t \rightarrow \infty$.

The KdV equation also allows other progressing singular solutions of the following kind,

$$u = \frac{2}{(x - x_1 - ct)^2} - \frac{c}{6}, \quad p = -\frac{2}{x - x_1 - ct} - \frac{c}{6} \left(x - \bar{x}_1 - \frac{1}{2} ct \right), \quad (10.9)$$

where c, x_1, \bar{x}_1 are constants. The jump-defect will also affect these in the following manner. Let q be given by

$$q = -\frac{2}{x - x_2 - ct} - \frac{c}{6} \left(x - \bar{x}_2 - \frac{1}{2} ct \right), \quad (10.10)$$

where x_2, \bar{x}_2 are two additional constants. Then the defect conditions require

$$\bar{x}_1 - \bar{x}_2 = -\frac{12}{c(x_1 - x_2)}, \quad \alpha = -\left(\frac{c}{6} + \frac{1}{(x_1 - x_2)^2} \right). \quad (10.11)$$

Again, as α becomes large $x_1 \rightarrow x_2$ and the effect of the defect on the singular solutions u and v disappears. However, the non-locality will require a large value of $\bar{x}_1 - \bar{x}_2$.

11. Discussion

The purpose of this investigation has been to discover to what extent the properties of jump-defects, originally described in certain relativistic integrable field theories, extend to non-relativistic systems of various kinds. In all cases, the defects are purely transmitting, in the sense that solitons passing through them may be delayed, converted to anti-solitons in some cases, or absorbed, but will never be reflected. In every case

examined, the integrable defect conditions investigated would constitute a Bäcklund transformation except that the spatial derivatives of the fields are frozen at the defect location. It is known that Bäcklund transformations for a given model are not unique and, particularly for the KdV and mKdV models, several different expressions are available in the literature. As Bäcklund transformations, all the available expressions are equivalent; but, as defect conditions, this does not appear always to be the case. For instance, for both the KdV and the mKdV equations local expressions, using u instead of $p = u_x$, for a Bäcklund transformation are available [18]. However, it does not seem that all formulations of Bäcklund transformations may be used as defect conditions since it is not always possible to give a Lagrangian description for the u -expressions for these models, and consequently to find suitable defect potentials. Moreover, to allow momentum conservation, which is a key feature of the jump-defect, the corresponding defect conditions for KdV and mKdV need to satisfy the additional relation listed at the end of section 9. Not all Bäcklund transformations seem to satisfy such a condition - for an example of one that does not, consider the Bäcklund transformation proposed in [21] for the KdV equation.

It ought to be possible to modify appropriate Lax pairs for KdV and mKdV [24] to accommodate the jump-defects. However, because of the proliferation of higher derivatives in the expressions for the Bäcklund transformations this appears to be less than straightforward and discussions of these will be deferred.

The nonlinear Schrödinger model seems to provide the nicest results. In this case, bound states have been found and it has been demonstrated using the two-soliton example that several solitons passing a defect will be affected by it independently of one another. If there are several jump-defects at different locations they too will act independently. It transpires that the strangest situation is illustrated by the KdV example. First, it does not seem possible to recover the single bulk soliton by taking a suitable limit of the defect parameter, and second the consequences of taking the limit $|\beta| \rightarrow 0$ remains mysterious. In fact, the situation in which $|\beta| < a$, that is the defect parameter is less than the soliton speed, does not seem to be allowed since the incoming soliton at the left of the defect would become singular on the right of it. One resolution of this that has been suggested is that the soliton is trapped by the defect (as is clearly the case when $|\beta| = a$).

It was noted in [13] that the jump-defects of the sine-Gordon model may themselves move and indeed scatter if there are several moving with different speeds. This aspect has not been considered in the present study though similar properties are expected, at least for NLS. The non-local nature of the defect conditions for the other models (KdV and mKdV) may pose a problem in this regard. It was also noted in [13] that the natural solution of the ‘triangle relations’ [14],

$$S_{kl}^{mn}(\Theta) {}^eT_{n\alpha}^{t\beta}(\theta_1) {}^eT_{m\beta}^{s\gamma}(\theta_2) = {}^eT_{l\alpha}^{n\beta}(\theta_2) {}^eT_{k\beta}^{m\gamma}(\theta_1) S_{mn}^{st}(\Theta), \quad \Theta = \theta_1 - \theta_2, \quad (11.1)$$

where the roman labels are ± 1 , corresponding to soliton or anti-soliton, the greek labels are even integers, corresponding to the ‘topological charge’ carried by the defect, $S_{kl}^{mn}(\Theta)$

is the bulk scattering matrix for a pair of solitons, and ${}^eT_{n\alpha}^{t\beta}(\theta)$ is the transmission matrix for a soliton passing through an even-charged defect, is consistent with the Lagrangian description of the sine-Gordon jump-defect. In the present context, the NLS model appears to be the most suitable for a quantum investigation and also the most interesting due to the presence of bound states. It is well known that the quantum version of NLS is equivalent to a one dimensional multi-particle problem with δ -function pairwise interactions of equal strength. The problem in the bulk has been studied long ago, in the first place by means of the coordinate Bethe ansatz [27] (but see also [28] for a review and more references). It is therefore natural to try to incorporate the jump-defect within the N body quantum picture and to discover what manner of particle interaction is able to describe it. More recently, an N body problem of this kind with impurity (in a ‘repulsive’ regime where there would be no solitons in the corresponding NLS model), has been analysed in [29]. However, more investigations are necessary to see if there are any connections between that type of impurity and the jump-defect described in this article.

One aspect of the story that remains frustrating is the absence of a physical model of an integrable jump-defect. It would be very interesting to discover a physical situation where the Bäcklund transformation plays a natural role. There are jump-defects commonly occurring naturally - modelling a dislocation in a material provides an example, as does modelling a shock front, or a fluid bore - where certain physical quantities are regarded as discontinuous, such as the fluid velocity on either side of the bore, yet others are continuous; and some conservation laws are preserved. However, there does not yet appear to be a nonlinear, and integrable, example of the kind we are discussing. If there were, it might offer the possibility of controlling solitons, which might then be put to use (see [30] for a suggestion).

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